DYNAMIC LOT-SIZING PROBLEMS: A Review on Model and Efficient Algorithm

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Abstract

Due to their importance in industry, dynamic demand lot-sizing problems are frequently studied. This study consider dynamic lot-sizing problems with recent advances in problem and model formulation, and algorithms that enable large-scale problems to be effectively solved. Comprehensive review is given on model formulation of dynamic lot-sizing problems, especially on capacitated lot-sizing (CLS) problem and the coordinated lot-sizing problem. Both approaches have their intercorrelated, where CLS can be employed for single or multi level/stage, item, and some restrictions. When a need for joint setup replenishment exists, then the coordinated lot-sizing is the choice. Furthermore, both algorithmics and heuristics solution in the research of dynamic lot sizing are considered, followed by an illustration to provide an efficient algorithm.

***Keyword**: Dynamic lot sizing, modeling, algorithm, heuristics.*

1. INTRODUCTION

Lot sizing problems are production planning problems with setups between production lots. By reason of these setups, it is often too costly to produce a given product in every period. On the other hand, generating fewer setups by producing large quantities to satisfy future demands results in high inventory holding costs. Thus, the objective is to determine the periods where production should occur, and the quantities to be produced, in order to satisfy demand while minimizing production, setup and inventory holding costs.

Lot-sizing problems have been studied extensively for the past half century. Wagner and Whitin (1958) propose a forward algorithm for a general dynamic version of the uncapacitated economic lot-sizing model. Since then, various variants, including single-item and multi-item, uncapacitated and capacitated lot-sizing problems, remain an important topic in Operations Research fields. However, the uncapacitated lot-sizing problem is an ideal case and hardly applicable to real-world operations. Furthermore, the general capacitated lot-sizing problem is NP-hard (see Bitran and Yanasse, 1981)

Capacitated dynamic lot sizing deals with the problem of determining time-phased production quantities that meet both given external demands and given capacity limits of the production system. The problem arises in production environments where the changeover of a resource from one product type to another causes setup time and/or setup costs.

For the (single-level) capacitated lot sizing problem (CLSP) and the multi-level capacitated lot sizing problem (MLCLSP), the problem is to determine production quantities and periods only, without consideration of the actual production sequence of the orders within a time period. This type of modeling has the advantage that it allows a flexible resequencing of orders within a period, at predetermined cost. However, a detailed production plan must be generated in a subsequent planning step. (Sahling *et al.*, 2009)

When concerned to the dynamic demand, the coordinated lot-size problem is the choice. It determines the time-phased replenishment schedule (i.e., timing and order quantity) that minimizes the sum of inventory and ordering costs for a family of items. A joint shared fixed setup cost is incurred each time one or more items of the product family are replenished, and a minor setup cost is charged for each item replenished. In addition, a unit cost is applied to each item ordered (see

Robinson *et al.*, 2009). Coordinated lot-size problems are often encountered in production, procurement, and transportation planning

The purpose of this paper is to address a review of the concepts of dynamic lot sizing. Throughout the paper, three relatively straightforward of lot sizing concepts are illustrated, namely: Uncapacitated Single Item Lot Sizing Problem (USILSP), Capacitated Lot Sizing Problem (CLSP), and the Coordinated-Capacitated Lot Sizing Problem (CCLSP). The basic concepts of lot sizing problem is illustrated in Section 2, while the concepts of USILSP, CLSP and CCLSP are described intensively together with the problem extension and variants in section 3. Section 4 explains recent trends in algorithm approaching of lot sizing problems briefly, enriched by an illustration of lot sizing algorithm for a real case of production, and compilation of several heuristics based on designated problem of lot sizing.

2. GENERAL OVERVIEW OF LOT-SIZING PROBLEMS

A variety of taxonomies are proposed for classifying lot-sizing problems. An important problem characteristic is the nature of demand. Static demand problems assume a stationary or constant demand pattern, while dynamic demand problems permit demand to vary. If all demand values are known for the duration of the planning horizon, the demand stream is defined as deterministic. Otherwise, the demand is considered to be stochastic.

The complexity of lot sizing problems depends on the features taken into account by the model. Karimi *et al.* (2003) explained that the characteristics affect the classifying, modelling and the complexity of lot sizing decisions including: (a) planning horizon, which is the time interval on which the master production schedule extends into the future, (b) number of levels, whether single-level or multi-level, (c) number of products, (d) capacity or resource constraints, include manpower, equipment, machines, budget, etc., (e) deterioration of items that influence the restrictions in the inventory holding time, (f) demand, and (g) setup structure.

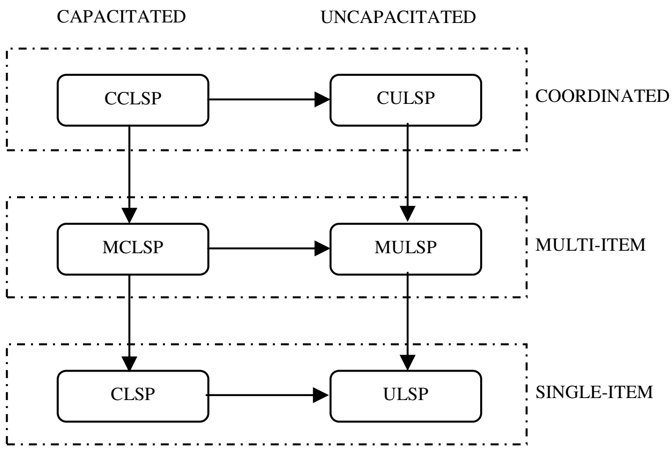

Figure 1. Taxonomy of deterministic dynamic demand lot-sizing problems (Robinson, 2009)

Fig 1. presents the six most commonly researched deterministic dynamic demand lot-sizing models. The problems are classified according to three factors: (1) single or multiple items, (2) capacitated or uncapacitated replenishment quantities, and (3) joint or independent setup cost structures. The problem classes are represented by nodes and their structural relationships by arcs, where a problem node originating an arc is a generalization of the problem node terminating the arc.

3. THE CONCEPTS OF LOT SIZING PROBLEMS

Lot sizing problems are production planning problems with setups between production lots. Because of these setups, it is often too costly to produce a given product in every period. On the other hand, generating fewer setups by producing large quantities to satisfy future demands results in high inventory holding costs. Thus, the objective is to determine the periods where production should take place, and the quantities to be produced, in order to satisfy demand while minimizing production, setup and inventory holding costs. Other costs might also be considered. Examples are backorder cost, changeover cost, etc.

3.1. Uncapacitated Single Item Lot Sizing Problem (USILSP)

The USILSP considers a single (or aggregate) product, and the production capacity is assumed to be high enough to never be binding in an optimal solution. USILSP is motivated by a possibility to aggregate products to obtain a single product (for example products which differ only in color can be considered as a single product) and where capacity is not a big concern. (Brahimi *et al.*, 2006).

USILSP is very often solved as a sub-problem in several algorithms for more complex lot sizing problems. These are basically presented as LP/IP models. For this reason, one must study the different IP formulations of the USILSP.

Let T be the length of the planning horizon and d_t, p_t, s_t, and h_t be the demand, unit production cost, setup cost and inventory holding cost, respectively, in period t $(t=1,\ldots,T)$. The decision variables are: X_t, the quantity to be produced in period t; It, the inventory level at the end of period t; and $Y_t=1$ if a setup occurs in period t $(X_t > 0)$ and zero otherwise. Also, let $d_{qt}=d_q+d_{q+1}+\ldots+d_t$.

A natural formulation of the problem is the following:

$$Minimize \quad \sum_{t=1}^{T}(s_tY_t + p_tX_t + h_tI_t), \qquad (1)$$

subject to :

$$I_{t-1} + X_t = d_t + I_t \qquad \forall t, \qquad (2)$$

$$X_t \le Y_t d_{tT} \qquad \forall t, \qquad (3)$$

$$Y_t = 0 \; or \; 1 \qquad \forall t, \qquad (4)$$

$$I_t, X_t \ge 0 \qquad \forall t, \qquad (5)$$

Assuming that without loss of generality, the stock at the beginning and the stock at the end of the planning horizon are zero. The objective function (1) is to minimizes the sum of setup, production and inventory holding costs over the whole N-period horizon. Constraints (2) are the inventory balance equations. They express that the entering stock (I_{t-1}) added to the current period production (X_t) are used to satisfy the demand (d_t). What remains is kept in stock at the end of the period (I_t). Constraints (3) relate the continuous production variables X_t to the binary setup variables Y_t.

3.1.1. *Extensions of the standard problem of USILSP*

Besides the consideration of capacity limits, several other extensions of the USILSP have been studied in the literature. These include, among others, backlogging, multiple facilities, remanufacturing and time windows.

Backlogging

If backlogging is allowed, a stockout cost is incurred for each unit backordered per time unit. This policy can be incorporated into USILSP formulation as follows:

$$Minimize \quad \sum_{t=1}^{T}(s_tY_t + p_tX_t + h_tI_t^+ + b_tI_t^-), \qquad (6)$$

subject to :

$$I_{t-1}^+ - I_{t-1}^- + X_t = d_t + I_t^+ - I_t^- \qquad \forall t, \qquad (7)$$

$$X_t \le Y_t d_{tT} \qquad \forall t, \qquad (8)$$

$$Y_t = 0 \; or \; 1 \qquad \forall t, \qquad (9)$$

$$I_t^+, I_t^-, X_t \ge 0 \qquad \forall t, \qquad (10)$$

In this model, the additional parameter is b_t which corresponds to the backlogging cost per period t. I_t is replaced by I_t^+, the number of items held in stock until the end of period t, to distinguish it from I_t^-, the quantity backlogged at the end of period t. The inventory balance constraints (7) and the non-negativity constraints (10) are modified to consider the new variables.

Remanufacturing

The lot-sizing problem with a remanufacturing option is an extension of the classical Wagner–Whitin model. The additional feature is that in each period a deterministic amount of returned items (returns for short) enters the system. These returns can be remanufactured to satisfy demand besides regular manufacturing. This means that there are two types of inventory: the inventory of returns and the inventory of serviceables, where a serviceable is either a newly manufactured item or a remanufactured returned item. (Van den Heuvel and Wagelmans, 2008)

Wang *et al.* (2011), addressed the single-item, dynamic lot-sizing problem for systems with remanufacturing and outsourcing. Therein, demand and return amounts are deterministic over a finite planning horizon. Demand may be satisfied by the manufacturing of new items, remanufactured items, or outsourcing, but it cannot be backlogged. The objective is to determine the lot sizes for manufacturing, remanufacturing, and outsourcing that minimize the total cost, which consists of the holding costs for returns and manufactured/remanufactured products, setup costs, and outsourcing costs.

Multiple facilities

The idea of multiple facilities was also introduced by Zangwill (1969) who considered two cases: facilities in parallel and facilities in series. In the case of parallel facilities, there is no interaction between the facilities and each facility satisfies its own requirements. In the case of serial facilities, the output from one facility becomes the input to another facility and the last facility satisfies the demand. In the USILSP with multiple facilities, the different setup, production and inventory costs might differ from one facility to another. Essentially, what makes the problem particular with respect to the original USILSP is the additional transfer variables W_{jkt} which represent the quantities to be transferred from plant j to plant k during period t. This problem can be represented by the following model:

$$\text{Minimize } \sum_j \sum_t (s_{jt} Y_{jt} + p_{jt} X_{jt} + h_{jt} I_{jt} + (\sum_{k \neq j} r_{jkt} W_{jkt})), \qquad (11)$$

subject to

$$I_{jt-1} + X_{jt} + \sum_{l \neq j} W_{ljt} = d_{jt} + I_{jt} + \sum_{l \neq j} W_{jkt} \qquad \forall j,t, \qquad (12)$$

$$X_{jt} \leq Y_t \sum_j \sum_{i=t}^{T} d_{ji} \qquad \forall j,t, \qquad (13)$$

$$Y_{jt} = 0 \; or \; 1 \qquad \forall j,t, \qquad (14)$$

$$I_{jt}, X_{jt}, W_{jkt} \geq 0 \qquad \forall j, k \neq j, t, \qquad (15)$$

$$I_{j0} = 0, I_{jt} = 0 \qquad \forall j \qquad (16)$$

In this model, transfer costs between plants are considered in the objective function. Inventory balance equations are modified to consider transferred products between plants.

Time windows

Time windows have been recently considered in the literature of lot-sizing. In Lee *et al.* (2001), the problem is based on demand time windows which are fixed by customers and considered as grace periods during which demand can be satisfied with no penalty; i.e. no inventory or backlogging costs are incurred when demands are completed within their time windows. Lee *et al.* assume special conditions on costs and study two cases: with and without backlogging. For the no-backlogging problem, an $O(T^2)$ algorithm is proposed. When backlogging is allowed, the problem is solved in $O(T^3)$.

Other extensions

The lot sizing model with cumulative capacities is an extension of dynamic lot sizing problem that concern to That each period has a production capacity, but unused capacity is transferred to the next period (Van den Heuvel and Wagelmans, 2008). This may be the case when capacity is not perishable, such as raw material or money. This is in contrast to the case of perishable capacity, such as time. Martel and Gascon (1995) introduce a dynamic lot sizing model with price changes and price-dependent holding costs. A dynamic programming approach is developed to solve it when solutions are restricted to sequential extreme flows, and results from location theory are used to derive an $O(T^2)$ algorithm which provides a provably optimal solution of an integer linear programming formulation of the general problem. Martel and Gascon also delivered a heuristic for the case where the inventory carrying rates and the order costs are constant, and where the item price can change once during the planning horizon. Permanent price increases, permanent price decreases and temporary price reductions are also considered.

3.2. Capacitated Dynamic Lot Sizing Problems

Capacitated dynamic lot sizing deals with the problem of determining time-phased production quantities that meet both given external demands and given capacity limits of the production system. The problem arises in production environments where the changeover of a resource from one product type to another causes setup time and/or setup costs. The CLSP is called a large bucket problem, because several items may be produced per period. Such a period typically represents a time slot of, say, one week in the real world. The planning horizon usually is less than six months. The decision variables for the CLSP are given in Table 1 while Table 2 provides the parameters.

Table 1. Decision Variable fo CLSP

Symbol	Definition
I_{jt}	Inventory for item j at the end of period t.
q_{jt}	Production quantity for item j in period t.
x_{jt}	Binary variable which indicates whether a setup for item j occurs in period t ($x_{jt} = 1$) or not ($x_{jt} = 0$).

Table 2. Parameters of CLSP

Symbol	Definition
C_t	Available capacity of the machine in period t.
d_{jt}	External demand for item j in period t.
h_j	Non-negative holding costs for item j.
I_{j0}	Initial inventory for item j.
J	Number of items.
p_j	Capacity needs for producing one unit of item j.
s_j	Non-negative setup costs for item j.
T	Number of periods.

$$Minimize \sum_{j=1}^{J}\sum_{t=1}^{T}(s_j x_{jt} + h_j I_{jt}) \qquad (11)$$

subject to

$$I_{jt} = I_{j(t-1)} + q_{jt} - d_{jt}, \qquad \forall j,t \qquad (12)$$

$$p_j q_{jt} \leq C_t x_{jt}, \qquad \forall j,t \qquad (13)$$

$$\sum_{j=1}^{J} p_j q_{jt} \leq Ct, \qquad \forall t \qquad (14)$$

$$x_{jt} \in \{0,1\}, \qquad \forall j,t \qquad (15)$$

$$I_{jt}, q_{jt} \geq 0, \qquad \forall j,t \qquad (16)$$

The objective (11) is to minimize the sum of setup and holding costs. Eq. (12) represents the inventory balances. Due to the restrictions (13), production of an item can only take place if the machine is set up for that particular item. Constraints (14) are the capacity constraints. The setup variables are defined to be binary (15) and the inequalities (16) are the nonnegativity conditions.

Solving the CLSP optimally is known to be NP-hard. If positive setup times are incorporated into the model, the feasibility problem is NP-complete. Here, there are only a few attempts to solve the CLSP optimally, many authors have developed heuristics.

3.2.1. *Variants of CLSP*

Most lot sizing problems are hard to solve. They prove that the single item capacitated problem is NP hard for quite general objective functions. Problems with concave cost functions and no capacity limits or constant capacities are solvable in polynomial time. Also lot sizing with convex cost functions and no set up cost is polynomially solvable (Jans and Degraeve, 2007).

Several variants of the CLSP covered include:

Time windows

Lee *et al.* (2001), were the first to study the uncapacitated problem with delivery time windows and discuss its applications in the context of supply chain collaboration. Under the assumption of non-speculative costs, they provided two optimal algorithms for the cases where backlogging is allowed and prohibited with computational complexities $O(T^3)$ and $O(T^2)$, respectively, where T is the length of the planning horizon. Hwang and Jaruphongsa (2006) extended these results to model speculative costs and provided an optimal algorithm that runs in $O(nT^3)$ time, where n is the number of demands.

Under capacity constraints, however, a demand may require several replenishments in multiple periods. That is, demand splitting cannot be avoided in the presence of production capacity. It is worth noting that, if demand-splitting is not allowed, the problem becomes NP-hard (reduction to a 2-Partition problem). (Hwang *et al.*, 2010). Furthermore, in the capacitated problem with delivery time windows (CP_TW), not only may a demand need to be split for production purposes, but, also, it is allowed to be released in multiple dispatches during the appropriate time window. The following parameters and decision variables for a mathematical formulation of the CP_TW are as follows:

Parameters

- C is the production capacity in each period.
- d_i is the required quantity for demand $i = 1,....,n$.
- K_t is the fixed setup cost of production in period $t = 1,....,T$.
- P_{it} is the unit production cost plus holding cost to satisfy
- demand i = 1,...,n by the production in period $t = 1,....,L_i$.

Decision variables

- y_{it} is the amount of demand $i = 1,...., n$ produced in period $t = 1,....,L_i$.
- z_t equals to 1 if there is a setup in period t, and 0 otherwise, for $t = 1,....,T$..

Formulation

$$\text{Minimize} \quad \sum_{t=1}^{T} K_t z_t + \sum_{i=1}^{n} \sum_{t=1}^{L_i} P_{it} y_{it} \qquad (17)$$

subject to :

$$\sum_{t=1}^{L_i} y_{it} = d_i \qquad i = 1,..., n \qquad (18)$$

$$\sum_{i:t \le L_i} y_{it} \le C z_t \qquad t = 1,..., T \qquad (19)$$

$$y_{it} \ge 0 \quad i = 1,..., n; \; t = 1,..., L_i \qquad (20)$$

$$z_t \in \{0,1\} \quad t = 1,..., T \qquad (21)$$

When there is sufficient capacity to accommodate all the demands, i.e. $\sum_{i:L_i \le t} d_i \le tC$ for all $t=1,...,T,$ this assumption guarantees a feasible solution to the CP with delivery time windows.

Lot sizing and scheduling problem

The other problem variants that can be identified together with an associated reference including: the economic lot scheduling problem (ELSP), the discrete lot sizing and scheduling problem (DLSP), the continuous setup lot sizing problem (CSLP), the proportional lot sizing and scheduling problem (PLSP). ELSP is a single-level, multi-item problem with stationary demand. The time is continuous and planning horizon is infinite. Solving the ELSP where capacity restrictions are involved is NP-hard (Li and Meissner, 2011). The NP-hard problem DLSP subdivides the (macro) periods of the CLSP into several (micro) periods. The fundamental assumption of the DLSP is the so-called all-or-nothing production, which means only one item may be produced per period, and, if so, the production amount would be as much as using full capacity (Karimi *et al.*, 2003).

The basic idea behind the PLSP (Drexl and Haase, 1995) is to use the remaining capacity for scheduling a second item in the particular period, if the capacity of a period is not used in full. The underlying assumption of the PLSP is that the setup state of the machine can be changed at most once per period. Production in a period could take place only if the machine is properly set up either at the beginning or at the end of the period. Hence, at most two products may be produced per period.

Some other variants based on restrictions

Sahling et al. (2009) proposed multi-level CLSP subjected to multi-period setup that carried over via a heuristic solution (MLCLSP-L). This model allows to carry over the setup state of a resource to the next periods following the setup. This leads to more efficient setup patterns and shorter planning-induced flow times. The solution was based on the fix-and-optimize heuristic.

In systems with a large demand size it is necesarry to consider a finite number of setups and inventory holding costs. Guu and Zhang (2002) introduced the multiple lot sizing problem in production systems with random process yield losses governed by the interrupted geometric (IG) distribution. This problem can be identified as an imperfect production process with known yield loss characteristics. Here, a dynamic upper bound on the optimal lot size is derived by an *O(nD)* algorithm, where *n* and *D* are the two-state variables. Furthermore, in the study of multiple lot-sizing problem with rigid demand, the cost structure and yield distribution are two main factors to determine the behavior of such problems. The decision problem is to select an initial production run size to minimize the expected total costs of possibly multiple runs for filing the demand. A dynamic programming can be applied to solve the problem (Guu, 1999).

Another complexity in lot sizing problem deal with in the numerous decisions a buyer has to make over time is the price increases or decreases to which many items are subjected. These price changes may take many forms such as a temporary low price, a permanent price increase, a permanent price decrease, etc., and are often known a few weeks in advance. Such changes must be carefully monitored and shrewdly acted upon by the buyer if she or he wants to minimize the total costs of acquiring, ordering and holding inventory. Here, Martel and Gascon (1997) proposed a dynamic programming to solve this problem due to a restriction of sequential extreme flows by deriving an $O(T^2)$ algorithm. A heuristic was also developed for the case where the inventory carrying rates and the order costs are constant, and where the item price can change once during the planning horizon. In another case, Li and Meissner, 2011, developed Mixed Integer Non- Linear Programming (MINLP) model for the cost minimizing problem when a capacity competition occured due to the complexity of time-varying demand with cost functions and economies of scale arising from dynamic lot-sizing costs. It is assumed that in the competition, each firm can replenish inventory at the beginning of each period in a finite planning horizon. Fixed as well as variable production costs incur for each production setup, along with inventory carrying costs. The individual production lots of each firm are limited by a constant capacity restriction, which is purchased up front for the planning horizon. The capacity can be purchased from a spot market, and the capacity acquisition cost fluctuates with the total capacity demand of all the competing firms.

Further, Pan *et al.* (2008) addressed the capacitated dynamic lot sizing problem arising in closed-loop supply chain where returned products are collected from customers. These returned products can either be disposed or be remanufactured to be sold as new ones again; hence the market demands can be satisfied by either newly produced products or remanufactured ones. The capacities of production, disposal and remanufacturing are limited, and backlogging is not allowed. A mode capacitated dynamic lot sizing problem with production, disposal and remanufacturing options is proposed to give a good approximation for such requirements. While, for a production of multi product with the inventories are replenished jointly whenever a common batch production occurs, and the output of any production batch always produces each individual product along a fixed ratio, then it is become a dynamic lot sizing with a joint replenishment model. To solve such problem, suppose that there are multiple types of products sharing the same production process, and assume that each batch will generate the same number of "units" of all products, then the planning horizon can be based on multiple discrete time periods, where each period has a demand of each product and a known cost structure. The decision is the production quantity in each period and inventory levels of all products will increase by the same quantity due to the assumption of scaled demand (see Lu and Qi, 2011).

3.3. Coordinated Dynamic Demand Lot Sizing

The coordinated dynamic demand lot sizing problem is an extension work of Robinson *et al.* (2008). For dynamic demand, coordinated lot-size problem determines the time-phased replenishment schedule (i.e., timing and order quantity) that minimizes the sum of inventory and ordering costs for a family of items. A joint shared fixed setup cost is incurred each time one or more items of the product family are replenished, and a minor setup cost is charged for each item replenished. In addition, a unit cost is applied to each item ordered. Demand is assumed to be deterministic but dynamic over the planning horizon and must be met through current orders or inventory. Coordinated lot-size problems are often encountered in production, procurement, and transportation planning.

3.3.1. *Coordinated uncapacitated lot-sizing problem (CULSP)*

The CULSP's objective is to minimize total system costs, which includes a joint setup cost for each time period any item in the product family is replenished, an item setup cost for each item replenished in each time period and inventory costs. The joint setup cost complicates the solution of the CULSP, which is known to be *NP*-complete. Robinson *et al.* (2009) present the four most significant problem formulations: (a) Traditional (TRAD) product unit formulation, (b) Shortest path (SPATH) formulation, (c) Arborescent network (ARBNET) formulation, and (d) Exact requirements (EXREQ) formulation. Among them, EXREQ showed the superiority over ARBNET in term of CPU times. The formulation is as follows. Consider a *T*-period planning horizon. For $i = 1, \ldots, I$ and $t = 1, \ldots, T$, define, d_{it}, the demand for the item *i* in period *t*; s_{it}, setup cost for item *i* in period *t*; S_t, joint setup cost in period *t*; c_{it}, variable per unit cost for item *i* in period *t*; and h_{it}, the per unit inventory holding cost for item *i* in period *t*. The decision variables include: x_{it}, order size for item *i* in period *t*; I_{it}, ending inventory of item *i* in period *t*; $Y_{it}=1$ if item *i* is replenished in period *t* and $Z_t = 1$ if a joint setup occurs in period *t*. For a specified setting of the joint setup variables, the resulting ULSPs are easily solved as *I* independent shortest path problems. Consider *I* items over a *T*-period planning horizon with $T'=T+1$. The demand for item *i* in periods *t'* through $t-1$ is $D_{it't}$. The total cost of ordering $D_{it't}$ units in period *t'* and serving demand through period $t-1$ for item *i* is $C_{it't} = s_{it'} + c_{it'}D_{it't} + \sum_{r=t'+1}^{t-1} h_{i,t'-1}D_{it't}$. The decision variable $Y_{it't} = 1$ if $D_{it't}$ units of item i are ordered in period *t'*, 0 otherwise. And by adding a the binary decision variable $wit't = 1$if and only if a replenishment is scheduled in time *t'* to cover the demand for item *i* from period *t'* through period *t*, and define $C'_{it't} = s_{it'} + c_{it'}\sum_{r=t'}^{t} d_{ir} + \sum_{t'+1}^{t}\left(\sum_{k=t'}^{r-1} h_{ik}\right)d_{ir}$ as the sum of the item production and inventory costs associated with $w_{it't}$.

The EXREQ formulation is:

$$Minimize Z = \sum_{t'=1}^{T} S_{t'}Z_{t'} + \sum_{i=1}^{I}\sum_{t'=1}^{T}\sum_{t=t'}^{T} C_{it't}w_{it't} \quad (21)$$

$$\text{Subject to } \sum_{i=1}^{t'}\sum_{t=t'}^{T} w_{irt} = 1 \qquad (i = 1,\ldots,I,\ t' = 1,\ldots,T) \quad (22)$$

$$\sum_{t=t'}^{T} w_{it't} \le Z_{t'} \qquad (i = 1,\ldots, I,\ t' = 1,\ldots,T) \quad (23)$$

$$w_{it't} \in \{0\, or\, 1\} \quad (i = 1,\ldots, I,\ t' = 1,\ldots, T, t = 1,\ldots,T) \quad (24)$$

$$Z_{t'} \in \{0\, or\, 1\} \quad (t' = 1,\ldots, T) \quad (25)$$

3.3.2. *Coordinated capacitated lot-sizing problem (CCLSP)*

The CCLSP contains both the complicating constraints associated with capacitated replenishment and the joint setup decision variables resulting in a *NP*-complete problem. Four alternative mathematical formulations are proposed by Robinson, *et al.*, shown that the most effective solution is by EXREQ formulation. Item setup cost $s_{it'}$ (28) and decision variable $Y_{it'}$ (29) are introduced to decouple the item and family setup constraints of EXREQ formulation for uncapacitated problem. Define $w'_{it't}$ as the fraction of the total demand for item *i* from period *t'* to period *t* that is served from an order in period *t'* and $\overset{\frown}{C}_{it't}$ as the sum of the variable per unit order and inventory holding costs for producing item *i* in period *t'*, where $\overset{\frown}{C}_{it't} = \sum_{q=t'+1}^{t}(c_{it'} + \sum_{k=t'}^{q-1} h_{ik})d_{iq}$.

The EXREQ formulation for this problem is as follows:

$$Minimize\ Z = \sum_{t'=1}^{T} S_{t'} Z_{t'} + \sum_{t'=1}^{T}\sum_{i=1}^{I} s_{it'} Y_{it'} + \sum_{i=1}^{I}\sum_{t'=1}^{T}\sum_{t=t'}^{T} \overset{\frown}{C}_{it't}\, w_{it't} \tag{26}$$

$$\text{Subject to} \sum_{i=1}^{t'}\sum_{t=t'}^{T} w'_{irt} = 1 \qquad (i=1,..,I,\ t=1,..,T) \tag{27}$$

$$\sum_{t=t'}^{T} w'_{it't} \le Y_{it'} \qquad (i=1,..,\ I,\ t=1,..,T) \tag{28}$$

$$Y_{it'} \le Z_{t'} \qquad (i=1,..,\ I,\ t=1,..,T) \tag{29}$$

$$\sum_{i=1}^{I}\sum_{t=t'}^{T} w'_{it't}\left(\sum_{q=t'}^{t} d_{iq}\right) \le P_{t'} Z_{t'} \quad (t'=1,...,T) \tag{30}$$

$$\sum_{t'=1}^{q}\sum_{t=q+1}^{T}\sum_{i=1}^{I} w'_{it't}\left(\sum_{r=t'+1}^{t} d_{ir}\right) \ge I_q^0 \quad (q=1,...,T) \tag{31}$$

$$0 \le w'_{it't} \le 1 \quad (i=1,...,I, t'=1,...,T, t=1,...,T), \tag{32}$$

$$Y_{it'} \in \{0\ or\ 1\} \quad (i=1,...,I, t'=1,...,T) \tag{33}$$

$$Z_{t'} \in \{0\ or\ 1\} \quad (t'=1,..,\ T) \tag{34}$$

The two important modeling features are compact structure of constraint set (27), which insures that all demand is met, and the surrogate aggregate inventory constraint (31).

4. ALGORITHM, ILLUSTRATION AND HEURISTICS METHODS

4.1. Algorithm and Illustration

Proofs from complexity theory as well as computational experiments indicate that most lot sizing problems are hard to solve. However, various solution techniques have been used to solve them. For instance, meta-heuristics such as tabu search, genetic algorithms and simulated annealing, have become popular and efficient tools for solving hard combinatorial optimization problems. (Jans and Degraeve, 2007).

The single item capacitated problem is NP hard for quite general objective functions. Problems with concave cost functions and no capacity limits (Wagner and Whitin, 1958) or constant capacities are solvable in polynomial time. Also lot sizing with convex cost functions and no set up cost is polynomially solvable. They proposed a dynamic programming (DP) recursion for the single item uncapacitated lot sizing problem. They prove that there exists an optimal solution that satisfies the following property: $s_{t-1}x_t = 0, \forall t \in T$. (see Van den Heuvel and Wagelmans, 2005) This property implies that there exists an optimal solution in which one never produces in a period and at the same time has inventory coming in from the previous period. As a consequence, production in one period satisfies the demand for an integral number of consecutive periods.

Now, the attention is turn on the Wagner-Whitin (WW) algrorithm for a finite-horizon, discrete-time model with deterministic but non-stationary demand for a single product at a single stage as developed by Muckstadt and Sapra (2010). In a finite-horizon discrete-time model, the length of the planning horizon is finite and the order placement decisions are made at discrete intervals of time. Here, three types of costs considered in this environment: the fixed ordering cost, procurement cost that is incurred only when an order is placed, and holding cost that is charged every period in proportion to the amount of on-hand inventory at a period's end.

The WW algorithm is based on the assumption and notation as follows:

- K_t = *fixed ordering cost*
- h_t = *holding cost per unit per period*
- C_t is the per-unit purchasing cost → $C_t + h_t \ge C_{t+1}$ for all t
- d_t is the known deterministic demand in period t
- x_t to represent the inventory at the beginning of period t before the order-placement decision is made
- For simplicity, assumed that lead time is zero
- y_t is the on-hand inventory after the order-placement decision is made and the order is received, or equivalent to x_t plus the order quantity. Here $y_t \ge x_t$
- The fixed cost is equal to K if $y_t > x_t$; otherwise it is 0.

Equivalently, the fixed cost as $K \cdot \delta_t$ where

$$\delta_t \begin{cases} 1, y_t > x_t \\ 0, \text{otherwise} \end{cases}$$

- The purchasing cost is equal to the product of the unit purchasing cost C and the order quantity $y_t - x_t$.
- Finally, the leftover inventory at the end of period t is $y_t - d_t$, and the corresponding holding cost is $h(y_t - d_t)$.

- Combining the three terms, the total cost in period t as $K\delta_t + C(y_t - x_t) + h(y_t - d_t)$

The algorithm is as follows:

- Define $F(t)$ to be the optimal cost from period 1 through period t when inventory at the end of period t is zero.
- Define ς_t^s to be the minimum cost over periods *1* through *t* when the inventory level at the end of period *t* is zero and period *t's* demand is satisfied by an order placed in period s.
- The optimal cost over periods *1* through *s−1* is equal to *F(s−1).*
- *The* cost incurred between periods *s* and *t* includes the fixed cost incurred in period s and the holding costs incurred in periods *s, s+1, . . . , t.*
- *T*he holding cost in period s is proportional to the inventory at the end of period *s,* which is equal to the sum of the demands in periods *s+1, s+2, . . . , t.*

The steps are like this:

- Step 1: Set *t = 2, v = 1 and F(1) = K.*
- Step 2: order is placed in period 1, determine whether to place an order in period 1 or in period 2 to satisfy period 2's demand. When the order is placed in period 2, the total cost is $F(1)+K_2 = K_1+K_2$ since no inventory is carried into period 2. When the order placed in period 1 is for d_1+d_2 units, the total cost is $\varsigma_2^1 = F(1) + hd_2 = K_1 + hd_2$ (4.1)
  - ✧ Choose the decision with the least total cost for periods 1 and 2. That is, $F(2) = \min\{K_1 + K_2, K_1 + hd_2\}$
  - ✧ and Set $v = 2$ if $K_1+K_2 < K_1+hd_2$. Otherwise, *v* remains unchanged.
- Step 3. Consider the t-period problem. Given *v,* the demand for period t is satisfied by placing the order in one of the periods *v, v + 1, v + 2, . . . , t.* Compute $\varsigma_t^v, \varsigma_t^{v+1}, ..., \varsigma_t^{t-1}, \varsigma_t^t$ using (4.1) and find $F(t) = \min\{\varsigma_t^v, \varsigma_t^{v+1}, ..., \varsigma_t^{t-1}, \varsigma_t^t\}$
- Step 4. Set $t \leftarrow t+1$. Stop if $t = T+1$. Otherwise, go to Step 3.

As an illustration, using a production data of MJOINT company – Yogyakarta, Indonesia (a producer of leather products), the working of the above algorithm is resulted as follows:

Table 3. Demand (order) of Leather Products - MJOINT co., 2011

Period (t) 2011	Leather Color (ft^2)				
	D. Brown	Black	Tobacco	Red	Aggregate
March	509	567	617	162	1855
April	1128	1063	816	409	3416
May	634	606	1029	217	2486
Junei	398	350	707	225	1680
July	505	577	825	288	2195
Total	3173	3162	3994	1301	11630

Known that, the ordering cost is Rp 1.750.000 (Rupiahs) and holding cost is Rp 700 per ft^2/month. The results are as follows:

Table 4. The Production Schedule for MJOINT case, 2011.

Period (t)	1	2	3	4	5
Demand (d_t)	1855	3416	2486	1680	2195
Order Quantity ($y_t - x_t$)	1855	3416	4166	1680	2195
Beginning of Period Inv. (x_t)	0	0	0	0	0
End of Period Inv. (x_{t+1})	0	0	1680	0	0
Cost in Period *t* (thousand rupiahs)	1.750	1.750	2.926	0	1.750

As seen on Table 4, the total cost as composed by order and holding cost for the whole periods is minimum as if the procurement designed for each period. This result is obtained based on the WW algorithm and it is effective in the situation that there is no capacity limitation and the demand is deterministic.

Most of algorithms in lot sizing have been developed based on their difference in the computational complexity. The WW algorithm may require a number of calculations proportional to T^2, where *T* is the length of the planning horizon. In some occasions it is named $O(n^2)$ or $O(T^2)$ algorithm. Another approach has been given by the Wagelmans–Hoesel–Kolen algorithm which requires calculations proportional to *T* log*T*. Hence this method is more efficient in finding the optimal solution. (Aggarwal and Park, 1992). Along with more restrictions to the problem, several algorithm is developed by giving more constraints respresenting those restrictions, such as in a reverse (product recovery) model, used products arrive to be stored and to be remanufactured at minimum cost. Richter and Sombrutzki (2000) present the WW algorithm applied to determine the periods in which used products are remanufactured or new products are produced. Also, the problem is extended to include additional variable manufacturing and remanufacturing cost (Richter and Weber, 2001). In fact, a generalized form of replenishment procedure it also includes backordering as well as replenishment cost that depend to the period in which the replenishment occurs. However, the algorithm has received extremely limited acceptance in practice (Silver and Peterson, 1985). In futher, the WW model could be represented as single source network since it is work out for single stage and item (Zangwill, 1969). When exposing it to the multi-stage again it can be represented as a decision network solution as developed by Gencer *et al.* (1998). For an $O(n^2)$ dynamic programming algorithm for lot sizing with inventory bounds and fixed costs, a solution can be obtained by utilizing a hierarchy of two layers of value functions (Alper and Simge, 2007)

When the problem faces a condition of limited capacity or capacitated problem, it can refer to an $O(T^3)$ algorithm and its extension as well. (see Okhrin and Richter, 2011). At last, as it develops, most discussed paper showed that an effective ways to solve kind of an extension in lot sizing problem is by employing heuristic methods.

4.2. Heuristics

Most well-known heuristics for lot-sizing problem are called the Silver–Meal and the Least Unit Cost heuristics. (see Ganas and Papachristos, 1997). Both heuristics are order T methods for computing a procurement plan. Although these approaches are very simple to implement, they do not necessarily obtain an optimal solution.

More efficient tools are meta-heuristics that including genetic algorithm, simulated annealing and tabu search. This heuristic is applicable to both uncapacitated or otherwise, including it extension as refer to single and multi item as a work of Jans and Degraeve (2006). The following table lists the major articles reviewed in this review and their associated research methodologies particularly in their proposed heuristics.

Table 5. Lot Sizing Problem and Heuristic Methods

Authors	Problem	Solution Method
Sahling, *et al.*, 2009	Multi-level capacitated with linked lot size	Fix-and-optimize algorithm
Heuvel and Wagelmans, 2008	The capacitated lot-sizing problem with linear costs	Dynamic programming
Xie and Dong, 2002	General capacitated lot sizing problem (and with overtiming)	Genetic algorithm
Gaafar, 2006	Dynamic lot sizing with batch ordering	Genetic algorithm compared to a modified Silver-Meal heuristic
Hop and Tabucanon, 2005	Lot-sizing problem with self-adjustment operation rate	Genetic algorithm
Eduardo and Barron, 2010	Lot-sizing problem with self-adjustment operation rate	A proof that Wagner Whitin method is more efficient than genetic algorithm developed by Hop and Tabucanon
Minner, 2009	Multi-product dynamic demand lot-sizing with limited warehouse capacity	Simple heuristics approach including constructive, smoothing and saving approach

Table 5. Lot Sizing Problem and Heuristic Methods (Continued)

Authors	Problem	Solution Method
Lee, *et al.*, 2005	Multi-product dynamic lot-sizing and shipping problem	Network model
Li, *et al.*, 2007	Capacitated production planning problems with batch processing and remanufacturing	Genetic algorithm
Narayanan and Robinson, 2010	Coordinated capacitated lot size problem	Six phase heuristic and simulated annealing meta-heuristic
Lyu and Lee, 2001	Dynamic lot sizing problem	Parallel algorithm

CONCLUSION

The numerous extensions of the basic lot sizing problem demonstrate that it can be used to model a variety of industrial problems. Lot sizing problems are challenging because many extensions are very difficult to solve. Several techniques have been discussed to tighten the model formulations. Some proofs reveal that the general algorithm of WW model has already led to promising a good result and enable to carry over some extensions. As shown on the result of MJOINT case, it is important to note that it will be more effective to determine a production schedule if it correspond to a dynamic demand but deterministic. Not only for obtaining a schedule but also more in its management aspects. Further, more opportunities for extending the WW model are still largely unexplored. The contribution of a heuristic method also should be appreciated as a comparison with other algorithms which may give more effective in problem structure.

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